4. Pringle, R. Jr., On the stability of a body with connected moving parts. AIAA Journal, Vol. 4, №8, 1966.
5. Whittaker, E. T., A treatise on the analytical dynamics of particles and rigid bodies. Cambridge Univ. Press, 1937.
6. Hooker, W. W. and Margulies, G., The dynamical attitude equations for an n-body satellite. J. Astronaut. Sci., Vol. 12, №4, 1965.
7. Roberson, R. E. and Wittenburg, J., Proc. of the 3rd Congress Internat. Federation of Automatic Control, London, 1966.
8. Hooker, W. W., A set of r dynamical attitude equations for an arbitrary n-body satellite having r rotational degrees of freedom. AIAA Journal, Vol. 8, №7, 1970.
9. Roberson, R. E. and Likins, P. W., A linearization tool for use with matrix formalism of rotational dynamics. Ingr-Arch., Bd. 37, Nr. 6, 1969.
10. Risito, C., On the Liapunov stability of system with known first integrals. Mechanica, Vol. 2, №4, 1967.
11. Magnus, K., Beiträge zur Dynamik des kräftefreien kardanisch gelagerten Kreisels. ZAMM, Bd. 35, Nr. 1, 1955.
12. Rumiantsev, V. V., On the stability of motion of a gyroscope on gimbals. PMM Vol. 22, №3, 1958.
13. Leimanis, E., The general problem of the motion of coupled rigid bodies about a fixed point. N.Y., Springer-Verlag, 1965.
14. Willems, P. Y., Attitude stability of deformable satellites. Evolut. attitude et stabilis. satellite, Colloq. internat. Paris, 1968.
15. Willems, P. Y., Stability of deformable gyrostats on a circular orbit. J. Astronaut. Sci., Vol. 18, №2, 1970.
16. Likins, P. W., Attitude stability criteria for dual-spin-spacecraft. J. Spacecraft and Rockets. Vol. 4, №12, 1967.

Translated by N. H. C.

UDC 531.36

ON BIFURCATION AND STABILITY OF STATIONARY MOTIONS IN CERTAIN PROBLEMS OF DYNAMICS OF A SOLID BODY

PMM Vol. 38, №4, 1974, pp. 616-627

V. N. RUBANOVSKII
(Moscow)

(Received December 19, 1973)

Bifurcation theory for stationary motions was developed by Poincaré [1] and Chetaev [2] for Lagrangian conservative mechanical systems. This theory is based on the investigation of the (transformed) potential energy of the system $V = V(c, q_1, \ldots, q_m)$, where $q_1, \ldots, q_m$ are the Lagrange coordinates and c is a parameter. For three problems in solid body dynamics we have shown below that this theory is applicable for the investigation of systems with known first integrals $U(x_1, \ldots, x_n) = c, \quad U_1(x_1, \ldots, x_n) = c_1, \ldots, U_k(x_1, \ldots, x_n) = c_k \ (k + 1 < n)$

As in the classical case, here we can introduce the function

$$W(c_1, \ldots, c_k; \ \lambda_1, \ldots, \lambda_k, x_1, \ldots, x_n) = U + \lambda_1 (U_1 - c_1) + \ldots + \lambda_k (U_k - c_k)$$

whose role is analogous to that of potential energy in the Poincaré-Chetaev theory; here $\lambda_1, \ldots, \lambda_k, x_1, \ldots, x_n$ formally play the role of the variables $q_1, \ldots, q_m$ $(k + n = m)$.

1. We have the equations

$$\partial W / \partial \lambda_1 = U_1 - c_1 = 0, \ldots, \quad \partial W / \partial \lambda_k = U_k - c_k = 0 \qquad (1.1)$$
$$\partial W / \partial x_1 = 0, \ldots, \quad \partial W / \partial x_n = 0$$

for defining the stationary motions. Let

$$\lambda_j = \lambda_j^\circ (c_1, \ldots, c_k), \quad x_i = x_i^\circ (c_1, \ldots, c_k) \qquad (1.2)$$
$$(j = 1, \ldots, k; \ i = 1, \ldots, n)$$

be a real solution of system (1.1). By Δ we denote the Hessian of the function W in the variables $\lambda_1, \ldots, \lambda_k, x_1, \ldots, x_n$, computed for solution (1.2). We have $\Delta = 0$ at the branch points of the solutions of Eqs. (1.1). Points for which $\Delta = 0$ are called bifurcation points. This definition coincides with the definition of bifurcation points for Lagrangian conservative mechanical systems [2].

Let $c_j = c_j(\alpha)$ $(j = 1, \ldots, k)$ be certain continuously-differentiable functions of parameter α. The expressions

$$\frac{d\lambda_j^\circ}{d\alpha} = \sum_{\nu=1}^{k} \frac{\partial \lambda_j^\circ}{\partial c_\nu} \frac{dc_\nu}{d\alpha} = \frac{\Delta^{(j)}}{\Delta}, \quad \frac{dx_i^\circ}{d\alpha} = \sum_{\nu=1}^{k} \frac{\partial x_i^\circ}{\partial c_\nu} \frac{dc_\nu}{d\alpha} = \frac{\Delta^{(k+i)}}{\Delta} \qquad (1.3)$$

hold for the derivatives $d\lambda_j^\circ / d\alpha$ and $dx_i^\circ / d\alpha$, where $\Delta^{(l)}$ denotes the determinant obtained from Δ by replacing its lth column by a column with the components $dc_1 / d\alpha, \ldots, dc_k / d\alpha, 0, \ldots, 0$. Relations (1.3) are analogs of relations well known in bifurcation theory (see the expression for δ on p. 42 of [2]); on this basis the analysis of the bifurcation points of the original system with many degrees of freedom can, under specific conditions, be reduced to the investigation of the bifurcation points of a certain reduced system with one degree of freedom, and *vice versa*.

We obtain the sufficient stability conditions for motion (1.2) from Routh's theorem as the sufficient conditions for the sign-definiteness of the quadratic form

$$\delta^2 W = \frac{1}{2} \sum_{i,j=1}^{n} \left(\frac{\partial^2 W}{\partial x_i \partial x_j} \right)_0 \delta x_i \delta x_j \qquad (1.4)$$

under the conditions

$$\delta U_j = \sum_{i=1}^{n} \left(\frac{\partial U_j}{\partial x_i} \right)_0 \delta x_i = 0 \quad (j = 1, \ldots, k), \qquad \det \left\| \left(\frac{\partial U_j}{\partial x_i} \right)_0 \right\|_1^k \neq 0 \quad (1.5)$$

These stability conditions can be represented in the form

$$D_\nu > 0 \quad \text{or} \quad (-1)^\nu D_\nu > 0 \qquad (\nu = 1, \ldots, n - k) \qquad (1.6)$$

where D_ν denotes the $(2k + \nu)$th-order principal diagonal minor of the determinant $D = (-1)^k \Delta = D_{n-k}$.

We examine the equation

$$\Delta(\varkappa) = \begin{vmatrix} \Theta & B \\ B' & A - \varkappa E_n \end{vmatrix}$$

where

$$A = \left\| \left(\frac{\partial^2 W}{\partial x_i \partial x_j} \right)_0 \right\|_1^n, \qquad B = \left\| \left(\frac{\partial U_j}{\partial x_i} \right)_0 \right\|_{i,\, j=1}^{j=k,\, i=n}$$

Θ is the $(k \times k)$ zero matrix, E_n is the nth-order unit matrix, and the prime denotes transposition. This equation is a natural generalization of the secular equation and has the real roots $\varkappa_1, \ldots, \varkappa_{n-k}$ which are the analogs of the Poincaré stability coefficients, while the number of negative roots is the analog of the degree of instability χ of motion (1.2) if $\Delta(0) = \Delta \neq 0$. When the system's motion is described by equations of the form

$$dx_i/dt = F_i(x_1, \ldots, x_n) \quad (i = 1, \ldots, n)$$

the following analog of the Kelvin-Chetaev theorem holds [2].

Theorem. If $D = (-1)^k \Delta < 0$, and if the equation

$$\left\| \left(\frac{\partial F_i}{\partial x_j} \right)_0 - \sigma\delta_{ij} \right\|_1^n = (-1)^n \sigma^{k+l} (\sigma^{2p} + g_1 \sigma^{2(p-1)} + \ldots + g_{p-1}\sigma^2 + g_p) = 0 \quad (1.7)$$

where $k + l + 2p = n,\ l \geqslant 0,\ \delta_{ij}$ are the Kronecker symbols, has no more than k zero roots, then the unperturbed motion (1.2) is unstable; here Eq. (1.7) has an odd number of positive roots.

The proof of this theorem is simple but cumbersome and is based on the investigation of the quadratic first integrals of the variational equations.

2. Let us examine the problem of the bifurcation and stability of the permanent rotations of a balanced gyrostat. Volterra [3] made a full investigation of these motions on the basis of a theorem which essentially is a modification of Routh's theorem. Rumiantsev [4] has investigated by Liapunov's direct method the stability of the permanent rotations of a gyrostat around its principal inertia axes which are possible under the condition that the gyrostatic moment vector is collinear with a principal inertia axis. However, the question of the bifurcation of these motions remains open.

The equations of inertial motion of a gyrostat with one fixed point O admit the integrals

$$U = \sum_{(123)} (J_1\omega_1 + g_1)^2 = \text{const}, \qquad U_1 = \sum_{(123)} J_1\omega_1^2 = h = \text{const}$$

Here $\omega_i,\ g_i\ (i = 1, 2, 3)$ are the projections of the gyrostat's absolute instantaneous angular velocity vector and of the gyrostatic moment vector onto the principal axes of its inertia ellipsoid for point O, J_i are the gyrostat's principal inertia moments; the summation sign with the symbol (123) denotes that two other terms are obtained by a cyclic permutation of the indices 1, 2, 3.

From equations of form (1.1) we find the following expressions for the projections of the angular velocity of the gyrostat's permanent rotations:

$$\omega_1 = \frac{g_1}{\lambda - J_1} \quad (1\,2\,3) \tag{2.1}$$

where $\lambda = \lambda(h)$ is an algebraic function of parameter h defined by the equation

$$h = \sum_{(123)} \frac{J_1 g_1^2}{(\lambda - J_1)^2} \tag{2.2}$$

The sufficient conditions (1.6) for the stability of motions (2.1) are reduced to the inequality

$$D = -\Delta = J_1J_2J_3(\lambda - J_1)(\lambda - J_2)(\lambda - J_3)\sum_{(1\,2\,3)}\frac{J_1\omega_1^2}{\lambda - J_1} > 0 \qquad (2.3)$$

By virtue of the above-cited theorem this condition is also a necessary stability condition. The first of formulas (1.3) leads to the relation

$$D = -\Delta = \frac{1}{2}J_1J_2J_3(J_1 - \lambda)(J_2 - \lambda)(J_3 - \lambda)\frac{dh}{d\lambda} \qquad (2.4)$$

from which it follows, in particular, that if $\lambda \neq J_1, J_2, J_3$, then $\Delta = 0$ if and only if $dh / d\lambda = 0$; therefore, at the bifurcation points, for which $\Delta = 0$, the function $h = h(\lambda)$ has stationary values.

The generators

$$\sum_{(1\,2\,3)} g_1(J_2 - J_3)\,\omega_2\omega_3 = 0 \qquad (2.5)$$

of a cone serve as the axes of the gyrostat's permanent rotations (2.1) in the body. The set of permanent rotations (2.1) can be represented geometrically as the curve $\lambda = \lambda(h)$ defined by Eq. (2.2). The correspondence between the points of curve $\lambda = \lambda(h)$ and the cone's generators (2.5) is established by formulas (2.1).

Figure 1 shows the graph of the function $h = h(\lambda)$ for the case when

$$(J_2 - J_3)(J_3 - J_1)(J_1 - J_2)\,g_1g_2g_3 \neq 0 \qquad (J_1 < J_2 < J_3) \qquad (2.6)$$

By $\lambda_j = \lambda_j(h)$ $(j = 1, \ldots, 6)$ we denote the branches of curve $\lambda = \lambda(h)$. As h varies within the intervals

$$0 < h < \infty, \quad \infty > h \geqslant h_{**}, \quad h_{**} \leqslant h < \infty, \quad \infty > h \geqslant h_*$$
$$h_* \leqslant h < \infty, \quad \infty > h > 0 \quad (h_* = h(\lambda_*),\ h_{**} = h(\lambda_{**}))$$

the values of λ_j range within the intervals

$$-\infty < \lambda_1 < J_1, \quad J_1 < \lambda_2 \leqslant \lambda_{**}, \quad \lambda_{**} \leqslant \lambda_3 < J_2$$
$$J_2 < \lambda_4 \leqslant \lambda_*, \quad \lambda_* \leqslant \lambda_5 < J_3, \quad J_3 < \lambda_6 < \infty$$

where λ_* and λ_{**} are roots of the equation $dh / d\lambda = 0$. By virtue of (2.1), to these six branches of curve $\lambda = \lambda(h)$ there correspond in the space $(\omega_1, \omega_2, \omega_3, h)$ six branches of the curve $\omega_i = \omega_i(h)$ $(i = 1, 2, 3)$; here the values $h = h_*$ and $h = h_{**}$ correspond to bifurcation points. These points are the limit points for those branches of curve $\omega_i = \omega_i(h)$ $(i = 1, 2, 3)$ which correspond to the branches λ_2, λ_3 and λ_4, λ_5 of curve $\lambda = \lambda(h)$. As h varies in the intervals $0 < h < h_{**}$, $h_{**} < h < h_*$ and $h_* < h < \infty$ we have, respectively, two, four and six branches of the curve $\omega_i = \omega_i(h)$ $(i = 1, 2, 3)$.

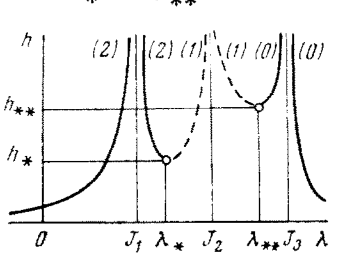

Fig. 1

The results of the investigation of stability condition (2.3) with the use of relation (2.4) are shown in Fig. 1 where the digits (0), (1), (2) on the branches of curve $\lambda = \lambda \times (h)$ indicate the degree of instability of motions (2.1), while bifurcation points correspond to the values $\lambda = \lambda_*$ and $\lambda = \lambda_{**}$. If condition (2.6) is not satisfied, the permanent rotations and their stability conditions can be obtained from (2.1)–(2.3) by appropriate passage to the limit.

For example, let $g_1 = 0$, $g_2 g_3 \neq 0$, $J_1 < J_2 < J_3$. Then

$$\omega_1 = \varkappa\delta(J_1 - \lambda), \quad \omega_2 = \frac{g_2}{\lambda - J_2}, \quad \omega_3 = \frac{g_3}{\lambda - J_3} \quad (\varkappa \text{ is a parameter}) \tag{2.7}$$

where $\delta(x) = 0$ if $x \neq 0$ and $\delta(0) = 1$. In space $(h, \lambda, \varkappa)$ the motions (2.7) can be represented geometrically as a surface $h = h(\lambda, \varkappa)$, whose equation is obtained as a result of substituting the values (2.7) into the energy integral $U_1 = h$. This surface consists of the cylindrical surface $h = h(\lambda, 0)$, $\lambda \neq J_1$ from which is removed the generator corresponding to the value $\lambda = J_1$, and of the parabola $h = h(J_1, \varkappa)$, $\lambda = J_1$, located in the plane $\lambda = J_1$. The cylindrical surface and the parabola have one common point for which $\lambda = J_1$, $\varkappa = 0$, $h = h(J_1, 0)$. Keeping in mind the fact that to each of the generators of the cylindrical surface there corresponds one and only one motion (2.7), we associate with the generators of this surface the points of their intersection with the plane $\varkappa = 0$ and for the geometric representation of motions (2.7) in the space $(h, \lambda, \varkappa)$, instead of the surface $h = h(\lambda, \varkappa)$ we examine the curve l whose individual branches lie in the orthogonal planes $\varkappa = 0$ and $\lambda = J_1$ and are specified by the equations $h = h(\lambda, 0)$, $\varkappa = 0$ and $h = h(J_1, \varkappa)$, $\lambda = J_1$, respectively.

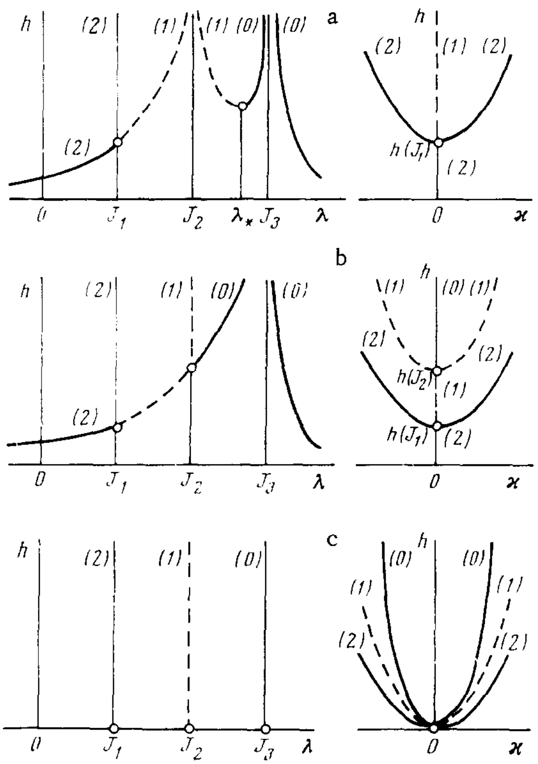

Fig. 2

The form of the projection of curve l onto the plane $\varkappa = 0$ is shown on the left in Fig. 2, a, while onto the plane $\lambda = J_1$ for $\lambda < J_2$ is shown on the right. Expression (2.4) for D takes the form

$$D = \frac{1}{2} J_1 J_2 J_3 (J_1 - \lambda)(J_2 - \lambda)(J_3 - \lambda) \frac{\partial h}{\partial \lambda}, \quad \text{if} \quad \lambda \neq J_1$$

$$D = \frac{1}{2} J_1 J_2 J_3 (J_2 - J_1)(J_3 - J_1) \varkappa \frac{\partial h}{\partial \varkappa}, \quad \text{if} \quad \lambda = J_1$$

Using these relations and the form of curve l we conclude that motions (2.7) are stable if $\lambda > J_3$, $\lambda_* < \lambda < J_3$ or $\lambda < J_1$, where λ_* is a root of the equation $\partial h / \partial \lambda = 0$ and are unstable if $J_1 < \lambda < J_2$ or $J_2 < \lambda < \lambda_*$; for $\lambda = J_1$ we have $D > 0$ for all $\varkappa \neq 0$, and motions (2.7) are stable. Bifurcation points correspond to the values $\lambda = \lambda_*$, $\varkappa = 0$ and $\lambda = J_1$, $\varkappa = 0$. In Fig. 2a, the digits (0), (1), (2) indicate the degree of instability of motions (2.7).

Now let $g_1 = g_2 = 0,\ g_3 \neq 0,\ J_1 < J_2 < J_3$. Then

$$\omega_1 = \varkappa\delta\,(J_1 - \lambda), \quad \omega_2 = \varkappa\delta\,(J_2 - \lambda), \quad \omega_3 = g_3/(\lambda - J_3) \tag{2.8}$$

The projections of curve l onto the plane $\varkappa = 0$ and onto the plane $\lambda = 0$ for $\lambda < J_3$ are shown in Fig. 2, b ; bifurcation points correspond to the values $\lambda = J_1,\ \varkappa = 0$ and $\lambda = J_2,\ \varkappa = 0$.

Finally, when $g_1 = g_2 = g_3 = 0,\ J_1 < J_2 < J_3$ (the Euler case), from (2.1) and (2.2) we obtain

$$\omega_1 = \varkappa\delta\,(J_1 - \lambda) \quad (1\,2\,3), \qquad h\,(\lambda, \varkappa) = \varkappa^2 \sum_{(123)} J_1\delta\,(J_1 - \lambda) \tag{2.9}$$

From stability condition (2.3) we conclude that the body's uniform rotations (2.9) around the minor and major axes of its inertia ellipsoid are stable, while the rotation around the middle axis is unstable. The projections of curve l onto the planes $\varkappa = 0$ and $\lambda = 0$ are shown in Fig. 2, c ; here all points of the λ-axis are bifurcation points and correspond to a neutral equilibrium position of the body.

3. Let us examine the problem of the motion in a central Newtonian force field of a solid body with one fixed point and with a cavity wholly filled with a homogeneous incompressible viscous liquid. We introduce a moving rectangular coordinate system $Ox_1x_2x_3$ with origin at the body's fixed point O at a distance R from the center of attraction N, and with axes coinciding with the principal axes of the system's inertia ellipsoid for point O. For simplicity of computation we assume that the principal axes of the liquid's inertia ellipsoid for point O coincide with the axes $x_1,\ x_2,\ x_3$. We introduce the notation: $A_i,\ B_i,\ C_i\ (i = 1, 2, 3)$ are the moments of inertia relative to the x_i-axis of the body, of the liquid and of the whole system, respectively ; $\omega_i,\ G_i,\ g_i$ are the projections onto the x_i-axis of the body's instantaneous angular velocity vector, of the liquid's kinetic moment vectors relative to point O in its absolute and relative motions, respectively; u_i are the projections onto the same axis of the relative velocity vector of the liquid particle with coordinates $x_1,\ x_2,\ x_3$; τ is the cavity volume ; ρ is the liquid density; μ is the coefficient of viscosity; g is the gravitational acceleration at a distance R from the center of attraction ; $\nu = 3gR^{-1}$; e_i are constants proportional to the x_i-axis projections of the vector from point O to the system's center of inertia ; γ_i are the direction cosines of the "vertical" NO relative to the x_i-axis.

The expressions for the kinetic and potential energies of the system have the form [4, 5]

$$T = \frac{1}{2} \sum_{(123)} (A_1\omega_1^2 + B_1^{-1}G_1^2 + w_1^2), \qquad \Pi = \frac{1}{2} \sum_{(123)} (\nu C_1\gamma_1^2 + 2e_1\gamma_1)$$

$$G_1 = B_1\omega_1 + g_1, \quad w_1^2 = \rho \int_\tau [u_1 + (\omega_2 - B_2^{-1}G_2)\,x_3 - (\omega_3 - B_3^{-1}G_3)\,x_2]^2\,d\tau\ (1\,2\,3)$$

The theorems of the kinetic energy and kinetic moment of the system lead to the relations

$$\frac{dU}{dt} = \frac{d}{dt}(T + \Pi) = -\mu \int_\tau \sum_{(123)} \left[2\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)^2\right] d\tau, \qquad U_1 = \sum_{(123)} (A_1\omega_1 + G_1)\,\gamma_1 = k = \text{const}$$

The values of variables $\omega_i,\ \gamma_i,\ G_i,\ w_i\ (i = 1, 2, 3)$ for which U has stationary values under the condition $U_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$ and for a given magnitude $U_1 = k$ of the area integral, correspond to permanent rotations of the whole system as one solid body around the vertical γ, defined by the formulas

$$\omega_1 = \omega\gamma_1,\ \Omega(C_1 - \lambda)\gamma_1 = e_1,\ G_1 = \omega B_1\gamma_1,\ w_1 = 0\ (u_1 = 0)\ (123) \qquad (3.1)$$

Here the dependences $\omega = \omega(k)$ of the angular velocity ω of the permanent rotation and of the intermediate parameter λ on the constant k are determined by the relations

$$\omega = \pm(\nu + \Omega)^{1/2}, \quad \Omega = \pm\left(\sum_{(123)} \frac{e_1^2}{(C_1 - \lambda)^2}\right)^{1/2}, \quad k = \frac{\omega}{\Omega^2}\sum_{(123)} \frac{C_1 e_1^2}{(C_1 - \lambda)^2} \qquad (3.2)$$

The sufficient stability conditions (1.6) for motions (3.1) with respect to the quantities $\omega_i, \gamma_i, G_i, w_i$ $(i = 1, 2, 3)$ when

$$\omega \sum_{(123)} (C_2 - C_3)^2 \gamma_2^2\gamma_3^2 \neq 0 \qquad (3.3)$$

reduce to the inequalities [5, 6]

$$D = \Delta = (4\omega^2 L + \Omega C S)\,\Omega > 0, \qquad \Delta_1 = \Omega L > 0 \qquad (3.4)$$

$$L = \sum_{(123)} (\lambda - C_1)(C_2 - C_3)^2\gamma_2^2\gamma_3^2, \quad S = \sum_{(123)} (\lambda - C_2)(\lambda - C_3)\gamma_1^2, \quad C = \sum_{(123)} C_1\gamma_1^2$$

If condition (3.3) is not satisfied, the second of inequalities (3.4) should be replaced by the following:

$$\Delta_1' = 4\omega^2 \sum_{(123)} (C_2 - C_3)^2\gamma_2^2\gamma_3^2 + C\Omega \sum_{(123)} (\lambda - C_1)(\gamma_2^2 + \gamma_3^2) > 0 \qquad (3.5)$$

If the sign of even one of inequalities (3.4) or (3.5) changes to the opposite one, the unperturbed motion (3.1) is unstable [4].

If there is no liquid in the body's cavity, then by virtue of the theorem cited in Sect. 1, motion (3.1) is unstable if the inequality in the first of conditions (3.4) changes sign. If, however, $\Delta > 0$ but $\Delta_1 < 0$ or $\Delta_1' < 0$, then to resolve the question of stability we should investigate the roots of the characteristic equation

$$\sigma^2(g_0\sigma^4 + g_1\sigma^2 + g_2) = 0 \qquad (3.6)$$

$$g_0 = C_1C_2C_3, \quad g_1 = \sum_{(123)} C_1\{(C_2 + C_3 - C_1)^2\omega^2 + [C_2(\lambda - C_2) + C_3(\lambda - C_3)]\,\Omega\}\gamma_1^2, \quad g_2 = \Delta$$

If even one of the inequalities

$$g_1 < 0, \quad g_2 < 0, \quad g_1^2 - 4g_0g_2 < 0 \qquad (3.7)$$

is satisfied, among the roots of Eq. (3.6) we can find a root with a positive real part, and motion (3.1) is unstable; here $\chi = 1$ if $g_2 < 0$, and $\chi = 2$ if $g_2 > 0$. However, if all the inequalities in (3.7) change sign, then motion (3.1) is stable in the first approximation and $\chi = 2$ if $\Delta_1 < 0$ or $\Delta_1' < 0$. In this case, if among motions (3.1) there are stable ones, then their stability bears a gyroscopic nature and collapses when the system is acted on by dissipative forces with complete dissipation (the latter occurs, for example, when a viscous liquid is present in the body's cavity).

4. The generators of the Shtoude cone in the body

$$\sum_{(1\,2\,3)} e_1 (C_2 - C_3)\gamma_2\gamma_3 = 0$$

serve as the axes of permanent rotations (3.1). The set of motions (3.1) can be represented geometrically in the form of the curve $\lambda = \lambda(k)$ defined by Eqs. (3.2). The relation between the points of curve $\lambda = \lambda(k)$ and the Shtoude cone generators is established by formulas (3.1).

From (1.3) we obtain the relation

$$\frac{dk}{d\lambda} = \frac{4\omega^2 L + \Omega CS}{2\omega(C_1 - \lambda)(C_2 - \lambda)(C_3 - \lambda)}$$

and we can represent the expression (3.4) for Δ as

$$\Delta = 2\omega\Omega(C_1 - \lambda)(C_2 - \lambda)(C_3 - \lambda)\, dk/d\lambda \qquad (4.1)$$

Hence it follows that if $\lambda \neq C_1, C_2, C_3$ and $\omega \neq 0$, then $\Delta = 0$ if and only if $dk/d\lambda = 0$. At the bifurcation points for which $\Delta = 0$, the tangent to curve $k = k(\lambda)$ is parallel to the λ-axis.

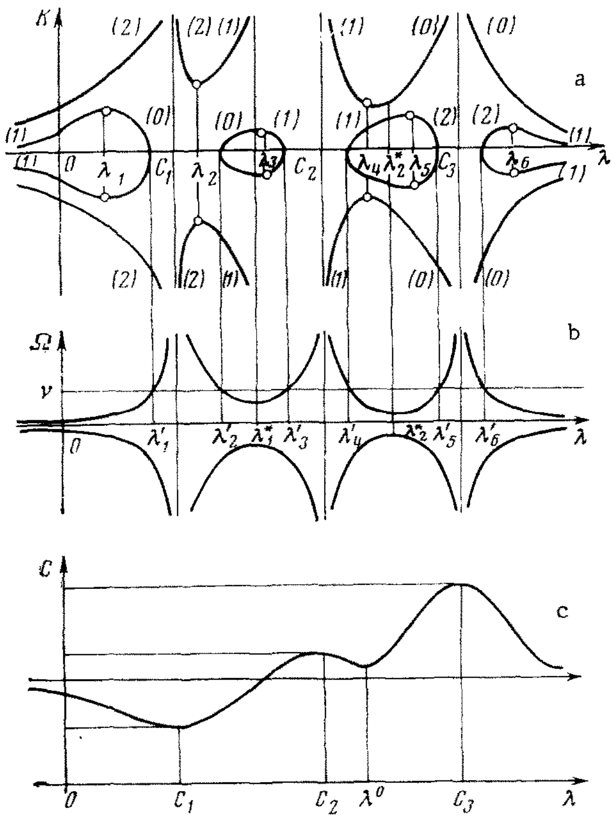

Fig. 3

Figure 3, a shows the graph of the function $k = k(\lambda)$ inverse with respect to the function $\lambda = \lambda(k)$ for the case when the equation $\Omega(\lambda) = \nu$ has six real roots λ_j' $(j = 1, \ldots, 6)$ and

$$(C_2 - C_3)(C_3 - C_1)(C_1 - C_2)\, e_1 e_2 e_3 \neq 0 \quad (C_1 < C_2 < C_3) \tag{4.2}$$

To analyze stability conditions (3.4) we use also the relations

$$\frac{d\Omega}{\partial\lambda} = \frac{\Omega S}{(C_1 - \lambda)(C_2 - \lambda)(C_3 - \lambda)}, \quad \frac{dC}{d\lambda} = \frac{2L}{(C_1 - \lambda)(C_2 - \lambda)(C_3 - \lambda)} \tag{4.3}$$

The second of these relations shows that if $\lambda \neq C_1, C_2, C_3$, then $dC / d\lambda = 0$ if and only if $L = 0$. Let us show that the equation $L(\lambda) = 0$ has only one real root $\lambda = \lambda^\circ$ and $C_1 < \lambda^\circ < C_3$. In fact, by substituting into expression (3.4) the values for γ_1, γ_2, γ_3 from (3.1) for L, we represent $L(\lambda)$ as

$$L(\lambda) = \frac{\Phi(\lambda)}{\Omega^4 (C_1 - \lambda)^2 (C_2 - \lambda)^2 (C_3 - \lambda)^2}$$

where

$$\Phi(\lambda) = \sum_{(123)} (\lambda - C_1)^3 (C_2 - C_3)^2 e_2^2 e_3^2 = a_0\lambda^3 - 3a_1\lambda^2 + 3a_2\lambda - a_3$$

$$a_m = \sum_{(123)} C_1^m (C_2 - C_3)^2 e_2^2 e_3^2 > 0 \quad (m = 0, 1, 2, 3)$$

Hence it follows that the sign of function $L(\lambda)$ coincides with the sign of $\Phi(\lambda)$; here $\Phi(\lambda) > 0$ if $\lambda > C_3$ and $\Phi(\lambda) < 0$ if $\lambda < C_1$. Further, the trinomial $d\Phi / d\lambda$ has complex roots since $a_1^2 - a_0 a_2 < 0$ by virtue of the Cauchy-Buniakowski inequality. Consequently, the equation $\Phi(\lambda) = 0$ has only one real root $\lambda = \lambda^\circ$; moreover, since

$$\Phi(C_2) = (C_1 - C_2)^2 (C_2 - C_3)^2 e_2^{\,2} [(C_2 - C_3) e_3^2 - (C_3 - C_2) e_1^2]$$

we have

$$C_2 < \lambda^\circ < C_3, \quad \text{if} \quad (C_2 - C_3) e_3^2 < (C_3 - C_2) e_1^2 \tag{4.4}$$

$$C_1 < \lambda^\circ < C_2, \quad \text{if} \quad (C_2 - C_3) e_3^2 > (C_3 - C_2) e_1^2$$

Figures 3b, c, show the graphs of the functions $\Omega = \Omega(\lambda)$ and $C = -C(\lambda)$ when conditions (4.2) and (4.4) hold. Here C_* denotes the system's moment of inertia relative to the straight line passing through the fixed point of the body and the system's center of inertia.

Note 1. For the geometric representation of motions (3.1) Kuz'min [6] used the function $\Omega = \Omega(\lambda)$ for the problem being considered here in the absence of the liquid. To study the Kharlamov's cone directrice of the axes of uniform rotations of a heavy gyrostat, the function $\omega = \omega(\lambda)$ was considered in [7, 8] (for the permanent rotations of a heavy solid body $\Omega(\lambda)$ and $\omega(\lambda)$ are related by $\Omega(\lambda) = \omega^2(\lambda)$). For this function there holds a relation analogous to the first one of (4.3); the latter relation was used to delineate a certain segment on the curve $\omega = \omega(\lambda)$ for which the stability conditions for the uniform rotations of a heavy gyrostat are satisfied. However, as seen from (3.4), (4.1) and (4.3), the critical points of the function $\omega = \omega(\lambda)$ for which $d\omega / d\lambda = 0$, are not bifurcation points in the sense adopted here and a change in the degree of instability of the unperturbed motions being investigated does not occur at these points.

5. By λ_j and λ'_j $(j = 1, \ldots, 6)$ we denote, respectively, the roots of the equations $dk / d\lambda = 0$ and $\Omega(\lambda) = \nu$, numbered so that the inequalities

$$\lambda_1 < \lambda'_1 < C_1 < \lambda_2 < \lambda_2' < \lambda_3 < \lambda_3' < C_2 < \lambda_4' < \lambda_4 < \lambda_5 < \lambda_5' < C_3 < \lambda'_6 < \lambda_6$$

hold in accordance with Fig. 3, a. Let us investigate the distribution of stable and unstable motions (3.1), at first on those branches of curve $k = k(\lambda)$ which do not intersect the λ-axis; for these branches $\Omega(\lambda) > 0$.

Let $\lambda > C_3$; then $L(\lambda) > 0$, $S(\lambda) > 0$, and conditions (3.4) are satisfied. Motions (3.1) are stable for $\lambda > C_3$ and $\chi = 0$. Now let $C_2 < \lambda < C_3$; then $\varphi(\lambda) = (C_1 - \lambda)(C_2 - \lambda)(C_3 - \lambda) > 0$, while $\omega(\lambda)$ and $dk / d\lambda$ (see Fig. 3, a) have values of the same sign if $\lambda_5 < \lambda < C_3$, and values of different signs if $C_2 < \lambda < \lambda_5$. By virtue of (4.1) we conclude from this that $\Delta > 0$ if $\lambda_5 < \lambda < C_3$, and $\Delta < 0$ if $C_2 < \lambda < \lambda_5$. Further, for $\lambda \approx C_3$ $(\lambda \neq C_3)$ we have $L(\lambda) > 0$ and conditions (3.4) are satisfied for such λ. Since the condition $\Delta > 0$ is the first to be violated [5] as λ varies continuously from the value for which conditions (3.4) are satisfied, it follows hence, firstly, that $\lambda^\circ \leqslant \lambda_5$ and, secondly, that motions (3.1) are stable $(\chi = 0)$ for $\lambda_5 < \lambda < C_3$, and unstable $(\chi = 1)$ for $C_2 < \lambda < \lambda_5$. Analogously, it is not difficult to show that motions (3.1) are unstable for $\lambda < C_2$ $(\lambda \neq C_1)$, where $\chi = 1$ if $\lambda_2 < \lambda < C_2$, and $\chi = 2$ if $C_1 < \lambda < \lambda_2$ or $\lambda < C_1$. In addition, the inequality $\lambda_2 \leqslant \lambda^\circ$ must hold.

We now consider the branches of curve $k = k(\lambda)$ which do intersect the λ-axis; for these branches $\Omega(\lambda) < 0$. Let $\lambda > \lambda_6'$; then $\Delta_1 = \Omega L < 0$, $\varphi(\lambda) < 0$, while $\omega(\lambda)$ and $dk / d\lambda$ have values of different signs if $\lambda > \lambda_6$ and values of the same sign if $\lambda_6' < \lambda < \lambda_6$. By virtue of (4.1) and (3.4) we conclude from this that motions (3.1) are unstable and $\chi = 1$ if $\lambda > \lambda_6$ and $\chi = 2$ if $\lambda_6' < \lambda < \lambda_6$. Analogous arguments lead to the following conclusions. Motions (3.1) are unstable if $\lambda_4' < \lambda < \lambda_5'$, $\lambda_3 < \lambda < \lambda_3'$ or $\lambda < \lambda_1$, and are stable if $\lambda_2' < \lambda < \lambda_3$ or $\lambda_1 < \lambda < \lambda_1'$.

The results of stability investigation of motions (3.1) are shown in Fig. 3, a where the values $\lambda = \lambda_j$ $(j = 1, \ldots, 6)$ correspond to bufurcation points. The distribution of the degree of instability on the branches of curve $\lambda = \lambda(k)$ holds also when there is no liquid in the body's cavity. In addition, in this case there exists $\lambda = \lambda_* < C_1$ such that motions (3.1) are stable in the first approximation if $\lambda_* < \lambda < \lambda_2$ $(\lambda \neq C_1)$, and unstable if $\lambda < \lambda_*$, on the branches of curve $\lambda = \lambda(k)$ for which $\Omega(\lambda) > 0$. We note that results analogous to the latter were obtained in [8] for the permanent rotations of a heavy gyrostat.

For fixed values of C_i, e_i $(i = 1, 2, 3)$, satisfying condition (4.2), and for a continuous variation of parameter ν from the values for which the equation $\Omega(\lambda) = \nu$ has six real roots to zero of the branch of curve $k = k(\lambda)$; they intersect the λ-axis, are deformed continuously so that the branches located between the values $\lambda = C_1, C_2$ and $\lambda = C_2, C_3$ shrink to a point lying on the λ-axis and vanish, while the branches for which $\lambda < C_1$ and $\lambda > C_3$ go off to infinity along the λ-axis: $\lambda_1' \to -\infty$, $\lambda_6' \to +\infty$ as $\nu \to 0$ $(\nu > 0)$. In the limit as $\nu \to 0$ $(\nu > 0)$ we obtain the geometric representation and the distribution of the degree of instability of the permanent rotations of a solid body with one fixed point in a uniform gravity field.

When condition (4.3) is not satisfied the investigation of the bifurcation and the stability of motions (3.1) can be carried out analogously as indicated in Sect. 2. For these cases the system's permanent rotations depend upon two parameters λ and $\varkappa$ and these motions can be represented geometrically in the space $(k, \lambda, \varkappa)$ as a surface $k = k(\lambda, \varkappa)$ whose equation can be obtained by substituting into the area integral $U_1 = k$ the values of $\omega_i, \gamma_i, G_i, w_i$ $(i = 1, 2, 3)$ corresponding to the permanent rotations being examined. The surface $k = k(\lambda, \varkappa)$ consists of a cylindrical surface $k = k(\lambda, 0)$, to each of whose generators corresponds one and only one permanent rotation, and of one, two or three curves located in the planes $\lambda = C_1, C_2, C_3$ and given by the equations $k = k(C_i, \varkappa)$, $\lambda = C_i$ $(i = 1, 2, 3)$.

Note 2. The problem on the motion of a solid body with one fixed point in a central Newtonian force field [9] is dynamically equivalent to the Brun problem and to the problem on the motion of a solid body (whose geometry and mass distribution satisfy the condition of the second integrable Clebsch case) in an unbounded ideal liquid under Chaplygin conditions, when the body's weight and the Archimedian propulsion force are equal in magnitude, while the impulsive force is vertical. Therefore, any result pertaining to one of these three problems can be extended to the other two. In particular, the geometric representation of the constant helical motions of the body in the liquid and the distribution of the degree of instability are the same as in Fig. 3, a ; the parameter k can be interpreted as the pitch of the body's helical motion.

As applied to the Brun problem the results obtained in Sects. 3–5 correspond to the attracting plane $(\nu > 0)$. The investigation of the bifurcation and the stability of the permanent rotations of a body for a repelling plane $(\nu < 0)$ can be carried out by analogy with Sects. 3–5. The investigation carried out shows that the branches of curve $k = k(\lambda)$ for which $\Omega(\lambda) < 0$ should be discarded since they correspond to purely imaginary values of parameter k, while the branches for which $\Omega(\lambda) > 0$ intersect the λ-axis at points determined from the equation $\Omega(\lambda) + \nu = 0$ and tend to the values $\lambda = C_1, C_2, C_3$ as $k \to \pm\infty$. The curve $k = k(\lambda)$ is symmetric relative to the λ-axis and the tangent to this curve at the bifurcation points is parallel to the λ-axis.

For example, if the equation $\Omega(\lambda) + \nu = 0$ has six real roots, then the curve $\lambda = \lambda(k)$ has six branches intersecting the λ-axis to the left and to the right of the values $\lambda = C_1, C_2, C_3$ and, under condition (4.3), bifurcation points do not exist on these branches since all motions (3.1) lying on any one of them possess one and the same degree of instability. If the equation $\Omega(\lambda) + \nu = 0$ has two roots $\lambda = \lambda_1'$ $(< C_1)$ and $\lambda = \lambda_2'$ $(> C_3)$, then the equation $dk / d\lambda = 0$ also has two roots $\lambda = \lambda_1$, $C_1 < \lambda_1 < C_2$, and $\lambda = \lambda_2$, $C_2 < \lambda_2 < C_3$. Motions (3.1) are stable $(\chi = 0)$ for $\lambda > \lambda_2$ $(\lambda \neq C_3)$, are unstable $(\chi = 1)$ for $\lambda_1 < \lambda < \lambda_2$, and the degree of instability $\chi = 2$ for $\lambda < \lambda_1$ $(\lambda \neq C_1)$. As $\nu \to 0$ $(\nu < 0)$ we have $\lambda_1' \to -\infty$, $\lambda_2' \to +\infty$ and in the limit we obtain the geometric representation and the distribution of the degree of instability of the permanent rotations of a heavy solid body.

REFERENCES

1. Poincaré, H., Sur l'equilibre d'une masse fluide animée d'un mouvement de rotation. Acta math., T.7, 1885.
2. Chetaev, N.G., Stability of Motion. Papers on Analytical Mechanics. Moscow, Izd. Akad. Nauk SSSR, 1962. (See also: (English translation) Chetaev, N.G., The Stability of Motion, Pergamon Press, Book № 09505, 1961).
3. Volterra, V., Sur la théorie des variations des latitudes. Acta math., T.22, 1889.

4. Moiseev, N. N. and Rumiantsev, V. V., Dynamics of a Body with Cavities Containing Liquid. Moscow, "Nauka", 1965.
5. Rubanovskii, V. N. and Stepanov, S. Ia., On the Routh theorem and the Chetaev method for constructing the Liapunov function from the integrals of the equations of motion. PMM Vol. 33, №5, 1969.
6. Kuz'min, P. A., Stationary motions of a solid body and their stability in a central gravity field. Tr. mezhvuz. konf. po prikl. teorii ustoichivosti i analit. mekhan., Kazan', 1964.
7. Kovalev, A. M. and Kiselev, A. M., On the cone of the axes of uniform rotation of a gyrostat. In: Mekhan. tverdogo tela, №4, Kiev, "Naukova Dumka", 1972.
8. Kovalev, A. M. and Kiselev, A. M., Delineation of a stability region on the cone of the axes of uniform rotation of a gyrostat. In: Mekhan. tverdogo tela, №4, Kiev, "Naukova Dumka", 1972.
9. Kharlamov, P. V., On the motion in a liquid of a body bounded by a multiply-connected surface. PMTF, №4, 1963.

Translated by N. H. C.

UDC 531.36

ON STABILIZATION OF THE ROTATIONAL MOTION OF A SOLID WITH FLYWHEELS IN A NEWTONIAN FORCE FIELD

PMM Vol. 38, №4, 1974, pp. 628-635
V. V. KREMENTULO
(Moscow)
(Received November 12, 1973)

A solution of the problem of optimal stabilization (in a specific sense) of the rotational motion of a gyrostat (a solid with two flywheels) in a central Newtonian force field is given within the framework of analytical control theory [1].

1. Initial equations of motion. Formulation of the problem. Retaining the notation used earlier [2], let us consider a solid along two of whose principal axes of inertia are located the axes of rotation of homogeneous symmetric flywheels, set in motion by special motors. The gyrostat is in a central Newtonian force field (O_1 is the attracting center, and O is the center of mass of the gyrostat).

Shown in Fig. 1 are the following coordinate systems: $O_1X_1X_2X_3$ – the inertial system, $Ox_1x_2x_3$ – rigidly coupled to the gyrostat and directed along its principal axes of inertia (Ox_1 and Ox_2 are the axes of flywheel rotation), $Ox_1'x_2'x_3'$ – semi-mobile (the Ox_3' axis coincides with the Ox_3 axis, while the Ox_1', Ox_2' axes do not take part in gyrostat rotation around the Ox_3 axis). Let us introduce the notation: C_1, C_2, C_3 are the gyrostat moments of inertia relative to the $Ox_1x_2x_3$ axes, respectively, J_1, J_2 are the axial moments of flywheel inertia (for a symmetric gyrostat $C_1 = C_2 = C$, $J_1 = J_2 = J$); q_1, q_2, q_3 are the projections of the instantaneous angular velocity of the trihedral $Ox_1'x_2'x_3'$ on these axes, β_{ik} are the direction cosines of the angles between the $O_1X_1X_2X_3$ and $Ox_1'x_2'x_3'$ axes, h_1, h_2, h_3 are projections of